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5ème Cours
Beijing
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The Curve

$$\begin{cases} E \rightsquigarrow [E:\mathbb{Q}_p] < \infty \\ E \rightsquigarrow \mathbb{F}_q((\pi)) \\ F|\mathbb{F}_q \text{ perfectoid} \end{cases}$$

$Y = \mathrm{Spa}(A, A) \setminus V(\pi[\varpi])$. $B = \mathcal{O}(Y)$

$\circlearrowright \varphi$

$|Y|^{cl} \subset |Y|$ classical Tate points

Def: $P = \bigoplus_{d \geq 0} \underbrace{B^{\varphi = \pi^d}}_{\{f \in B \mid \varphi(f) = \pi^d f\}}$ graded E-algebra.

Rem: * $B^{\varphi = Id} = E$. In fact for $f \in B \setminus \{0\}$, $\mathrm{Newt}(\varphi(f)) = \mathrm{Newt}(f)(q\cdot)$

$\varphi(f) = f \Rightarrow \mathrm{Newt}(f)(q\cdot) = \mathrm{Newt}(f) \Rightarrow$ only slope of $\mathrm{Newt}(f)$ is 0

$\Rightarrow \exists i \in \mathbb{Z}, \begin{cases} \mathrm{Newt}(f)_{|]-\infty, i[} = +\infty \\ \mathrm{Newt}(f)_{|[i, +\infty[} = cst \end{cases}$

[$\mathrm{Newt}(f)(x) \xrightarrow[x \to -\infty]{} +\infty$]

$\Rightarrow f \in E$

* Same method $\Rightarrow$ $B^{\varphi=\pi^d}=0$ if $d<0$.

Def: $X = \mathrm{Proj}(P)$ E-scheme

Th: (1) (The curve is a curve) X is a Dedekind scheme

(2) There is a morphism of ringed spaces

$$Y/\varphi^{\mathbb{Z}} \xrightarrow{\text{GAGA}} X$$

adic curve

inducing $|Y|^{cl}/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X|$

such that if $\underset{|Y|^{cl}}{\underset{\cap}{y}} \mapsto x$ then $\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{Y,y} = B^+_{dR,y}$

In particular residue fields at closed points of X are perfectoid fields $|E$, K, s.t. $[K^\flat : F] < +\infty$

(3) Set for $x \in |X|$ $\deg(x) := [k(x)^\flat : F]$

Then $\forall f \in E(X)^{\times}$ $\deg(\operatorname{div} f) = 0$ ②

(4) If F alg. closed, $\infty \in |X|$ then $\exists t \in H^0(X, \mathcal{O}(1))$,
$= B^{\varphi = \pi}$

$t \neq 0$, s.t. $\{\infty\} = V^+(t)$. Then if $B_e = B[\frac{1}{t}]^{\varphi = Id}$

$X \setminus \{\infty\} = \operatorname{Spec}(B_e)$

B_e is a P.I.D.

$(B_e, -\operatorname{ord}_\infty)$ non-euclidean almost euclidean

$\forall x, y \neq 0$, $x = ay + b$
$\deg b \leq \deg y$

$\rightarrow H^1(X, \mathcal{O}_X) = 0$
$H^1(X, \mathcal{O}_X(-1)) \neq 0$ Contrary to $\mathbb{P}^1$

Case F alg. closed

Th. : F alg. closed. P est gradué factoriel d'éléments irréductibles de degré 1 i.e. le monoïde abélien

$$\bigcup_{d\geq 0} P_d \setminus \{0\} / E^\times \text{ est libre sur } P_1 \setminus \{0\}/E^\times$$

En particulier $\forall x \in P_d \setminus \{0\}$, $d>0$, $\exists t_1, \ldots, t_d \in P_1$ s.t. $x = t_1 \cdots t_d$.

$\rightarrow$ Isomorphism of monoids

$$\bigcup_{d\geq 0} P_d \setminus \{0\}/E^\times \xrightarrow{\text{div}} \underbrace{\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})}_{\text{abelian free}} := \mathrm{Div}^+(Y)^{\varphi = Id}$$

on $|Y|^{cl}/\varphi^{\mathbb{Z}} \hookrightarrow \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$

$$y \bmod \varphi^{\mathbb{Z}} \longmapsto \sum_{n\in\mathbb{Z}} [\varphi^n(y)]$$

$\mathrm{div}(f) = \sum_{y \in |Y|^{cl}} \mathrm{ord}_y(f)[y]$, $\mathrm{ord}_y : B^+_{dR,y} \longrightarrow \mathbb{N} \cup \{+\infty\}$

$$\varphi(f) = \pi^d f \Rightarrow \underbrace{\mathrm{div}(\varphi(f))}_{\varphi^* \mathrm{div}(f)} = \underbrace{\mathrm{div}(\pi^d)}_{0} + \mathrm{div}(f)$$

$\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$ is a graded monoïd by setting

for $D = \sum_{y \in |Y|^{cl}} m_y [y] \in \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$

$\deg(D) = \sum_{y \in |Y_{[\rho^q, \rho[}|^{cl}} m_y$ for any $\rho \in]0,1[$

* Newton polygon consideration $\Rightarrow$ div is a graded morphism of graded monoids: $f \in B^{\varphi = \pi^d} \setminus \{0\} \Rightarrow \deg(\mathrm{div}\, f) = d$

* Injectivity: $f \in P_d \setminus \{0\}$, $g \in P_{d'} \setminus \{0\}$

$\mathrm{div}(f) = \mathrm{div}(g) \Rightarrow \deg(\mathrm{div}\, f) = \deg(\mathrm{div}\, g)$

$\Rightarrow d = d'$.

see preceding lectures $\longrightarrow$ $\Downarrow$

$f = u g$ with $u \in B^\times$

Moreover $\varphi(u) = u \Rightarrow u \in E^\times$.

* Surjectivity:

Weierstrass products:

$$y = (\underbrace{\pi - [z]}_{\xi}) \qquad 0 < |z| < 1$$

We are looking for $f \in \mathbb{P}_1 \setminus \{0\}$ s.t. $\operatorname{div}(f) = \sum_{n \in \mathbb{Z}} [\varphi^n(y)]$

$$\Pi^+(\xi) = \prod_{n \geq 0} \frac{\varphi^n(\xi)}{\pi} = \prod_{n \geq 0} \left(1 - \frac{[z^{q^n}]}{\pi}\right) \quad \text{C.V.}$$

$$\operatorname{div}(\Pi^+(\xi)) = \sum_{n \geq 0} [\varphi^n(y)]$$

Want to define $\Pi^-(\xi) = \text{``}\underbrace{\prod_{n < 0} \varphi^n(\xi)}_{\text{does not C.V.}}\text{''} \in A$ no renormalization process

Has to satisfy functional eq. $\varphi(\Pi^-(\xi)) = \xi . \Pi^-(\xi)$

<u>Prop:</u> Up to an element of $E^\times$, $\exists!\ \Pi^-(\xi) \in A \setminus \{0\}$ s.t.

$$\varphi(\Pi^-(\xi)) = \xi . \Pi^-(\xi)$$ → uses F alg. closed

→ Constructed by successive approximations.

by solving Kummer and then Artin–Schreier equations

④

$\rightarrow$ Want to define "$\prod_{n \leq 0} \varphi^n(x)$" for $x \in A$

modulo π, $x \equiv \alpha \in \mathbb{F}$

$$\text{"}\prod_{n \geq 0} \alpha^{q^n} = \alpha^{\frac{1}{q} + \frac{1}{q^2} + \dots} = \alpha^{\frac{1}{q-1}}\text{"}$$

$\rightsquigarrow$ need to solve Kummer equation $X^{q-1} - \alpha = 0$

well defined up to $\mathbb{F}_q^\times$

Suppose we have defined the product modulo π^b and want to define it modulo π^{b+1}.

$\rightarrow$ define "$\prod_{n \leq 0} \varphi^n(1 + \pi^b x)$" modulo $1 + \pi^{b+1} A$

$x \equiv \alpha \in \mathbb{F}$

need to define "$\sum_{n \leq 0} \alpha^{q^n} = \alpha^{\frac{1}{q}} + \alpha^{\frac{1}{q^2}} + \dots$"

Solution of Artin-Schreier $X^q - X = \alpha$

$\square$

Then $\varphi(\Pi^-(y)) = \xi \cdot \Pi^-(y)$

$\Rightarrow \operatorname{div}(\Pi^-(y)) = \sum_{n < 0} [\varphi^n(y)]$

$\Rightarrow \Pi(y) = \Pi^+(y) \cdot \Pi^-(y)$

$\operatorname{div}(\Pi(y)) = \sum_{n \in \mathbb{Z}} [\varphi^n(y)]$

Links with the logarithm

$E = \mathbb{Q}_p$. Idem everywhere with Lubin–Tate instead of $\widehat{\mathbb{G}}_m$.

<u>Recall:</u> $V = \widehat{\mathbb{G}}_m(\mathcal{O}_F) = (1+\mathfrak{m}_F, \times) \otimes$ $\mathbb{Q}_p$-vector space

$\varepsilon \in 1+\mathfrak{m}_F$, $\varepsilon \neq 1$, set $u_\varepsilon = \dfrac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} = 1+[\varepsilon^{1/p}]+\dots+[\varepsilon^{\frac{p-1}{p}}] \in A$

primitive degree 1

Then
$$\begin{aligned} V\setminus\{0\}/\mathbb{Z}_p^\times &\xrightarrow{\ \sim\ } |Y|^{cl} \\ \varepsilon^{\mathbb{Z}_p^\times} &\longmapsto V(u_\varepsilon) \end{aligned}$$

and if $C_\varepsilon = B/u_\varepsilon$ via $F \xrightarrow{\sim} C_\varepsilon^\flat$

$\varepsilon \mapsto$ generator of $\mathbb{Z}_p(1)$ in C_ε

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* Let us remark that $\Pi^+(\varphi_\xi) = \prod_{n\geq 0} \frac{\varphi^n(\xi)}{\pi}$

exists for ξ primitive of degree 1 satisfying

$$\xi \equiv \pi \mod W_{O_E}(m_F)$$

i.e. $W_{O_E}(O_F) \twoheadrightarrow W_{O_E}(k_F)$

$$a \longmapsto \pi$$

Since then $\frac{\xi}{\pi} - 1 \in W_{O_E}(m_F)$ and $\forall x \in W_{O_E}(m_F)$

$$\underbrace{\lim_{n\to+\infty} \varphi^n(x) = 0}$$

for the $(\pi, [\varpi])$-adic topology

In the same way $\Pi^-(\xi) \in A$ the unique solution

up to $E^\times$ of $\varphi(\Pi^-(\xi)) = \xi . \Pi^-(\xi)$ exists

* But $u_\varepsilon \equiv p \mod W(m_F)$

What is $\Pi^+(u_\varepsilon)$?

$$\Pi^+(u_\varepsilon) = \prod_{n\geq 0} \frac{\varphi^n(u_\varepsilon)}{p} = \lim_{n\to+\infty} \underbrace{\prod_{b=0}^{n} \frac{\varphi^b(u_\varepsilon)}{p}}$$

$$\frac{1}{p^{n+1}} \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} \times \frac{[\varepsilon^p]-1}{[\varepsilon]-1} \times \dots \times \frac{[\varepsilon^{p^n}]-1}{[\varepsilon^{p^{n-1}}]-1}$$

$$= \lim_{n\to+\infty} \frac{1}{p^{n+1}} \left(\frac{[\varepsilon^{p^n}]-1}{[\varepsilon^{1/p}]-1} \right)$$

$$= \frac{1}{p([\varepsilon^{1/p}]-1)} \lim_{n\to+\infty} \frac{1}{p^n} \left([\varepsilon^{p^n}] - 1 \right)$$

Recall: $\widehat{\mathbb{G}_m}$ $\quad X +_{\widehat{\mathbb{G}_m}} Y = XY + X + Y$

$$\log_{\widehat{\mathbb{G}_m}} = \log(1+T)$$

$$\log_{\widehat{\mathbb{G}_m}} = \lim_{n\to+\infty} \frac{[p^n]_{\widehat{\mathbb{G}_m}}}{p^n} = \lim_{n\to+\infty} \frac{(1+T)^{p^n}-1}{p^n}$$

as rigid analytic function on $\mathbb{D}_{\mathbb{C}_p} = \{|T|<1\}$

$$\Rightarrow \Pi^+(u_\varepsilon) = \frac{\log([\varepsilon])}{p([\varepsilon^{1/p}]-1)} \quad \text{in } B$$

Moreover $\varphi([\varepsilon^{1/p}]-1) = [\varepsilon]-1 = ([\varepsilon^{1/p}]-1)\cdot u_\varepsilon$

$\Rightarrow$ Can take $\Pi^-(u_\varepsilon) = p([\varepsilon^{1/p}]-1)$

$\Rightarrow \boxed{\Pi(u_\varepsilon) = \log([\varepsilon])}$

= Fontaine's t associated to ε as a generator of $\mathbb{Z}_p(1)$ in $\mathbb{C}_p^\flat$.

$$\operatorname{div}(t) = \sum_{n\in\mathbb{Z}} [\varphi^n(y)] \quad \forall\, y = V(u_\varepsilon)$$

$\Rightarrow \mathrm{ord}_y(t) = 1 \Rightarrow t$ uniformizing element of $B^+_{dR,y}$.

$\square$

<u>Corollary</u>: $\forall t \in P_1 \setminus \{0\}$, $P[\frac{1}{t}]_0 = B[\frac{1}{t}]^{\varphi = Id}$

is a factorial ring with irreducible elements

$\left\{ \frac{t'}{t} \;\middle|\; t' \in P_1 \setminus E^*.t \right\}$

<u>The fundamental exact sequence</u>

Fały-closed